

A parametrization of two-dimensional turbulence based on a maximum entropy production principle with a local conservation of energy

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Abstract. In the context of two-dimensional (2D) turbulence, we apply the maximum entropy production principle (MEPP) by enforcing a local conservation of energy. This leads to an equation for the vorticity distribution that conserves all the Casimirs, the energy, and that increases monotonically the mixing entropy (H -theorem). Furthermore, the equation for the coarse-grained vorticity dissipates monotonically all the generalized enstrophies. These equations may provide a parametrization of 2D turbulence. They do not generally relax towards the maximum entropy state. The vorticity current vanishes for any steady state of the 2D Euler equation. Interestingly, the equation for the coarse-grained vorticity obtained from the MEPP turns out to coincide, after some algebraic manipulations, with the one obtained with the anticipated vorticity method. This shows a connection between these two approaches when the conservation of energy is treated locally. Furthermore, the newly derived equation, which incorporates a diffusion term and a drift term, has a nice physical interpretation in terms of a selective decay principle. This gives a new light to both the MEPP and the anticipated vorticity method.

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1. Introduction

A remarkable property of two dimensional (2D) turbulent flows is their ability to organize spontaneously into coherent structures such as large-scale vortices and jets (Bouchet and Venaille 2012). A famous example of this self-organization is Jupiter’s Great Red Spot (GRS), a huge anticyclonic vortex persisting for more than three centuries in the southern hemisphere of the planet. This self-organization of 2D turbulence into large-scale vortices shares fascinating analogies with the self-organization of stellar systems in astrophysics (Chavanis et al. 1996, Chavanis 2002).

Basically, geophysical and astrophysical flows are described by the 2D Euler equations or by their generalizations (quasi-geostrophic equations, shallow-water equations, primitive equations...). The 2D Euler-Poisson system is known to develop a complicated mixing process generating vorticity filaments at smaller and smaller scales. At the same time, this mixing process leads to the formation of coherent structures at large scales which look quasistationary provided that a coarse-graining procedure is introduced to smooth-out the filaments. In order to explain this self-organization, a statistical theory of the 2D Euler equation has been proposed by Miller (1990) and Robert and Sommeria (1991). This statistical theory is the counterpart of the theory of violent relaxation proposed by Lynden-Bell (1967) for the Vlasov-Poisson system describing collisionless stellar systems such as elliptical galaxies. In the Miller-Robert-Sommeria (MRS) theory, the statistical equilibrium state of the 2D Euler equation (most probable or most mixed state) is obtained by maximizing a mixing entropy while conserving the energy and the infinite family of Casimirs.

Of course, the evolution of the coarse-grained vorticity $\overline{\omega}(\mathbf{r}, t)$ which averages over the filaments and which relaxes towards a quasistationary state is *not* given by the 2D Euler equation. We expect that it satisfies a kinetic equation that relaxes towards the maximum entropy state. Actually, the relaxation towards the maximum entropy state is not granted since it depends on an hypothesis of ergodicity (or at least efficient mixing) that is not always satisfied. Indeed, there are many situations in which the evolution of the system is non-ergodic so that the QSS differs from the statistical prediction.

An interesting problem is to determine the kinetic equation satisfied by the coarse-grained vorticity $\overline{\omega}(\mathbf{r}, t)$. This is interesting not only at a conceptual level, but also at a practical level. Indeed, it is generally not possible to solve the 2D Euler equations for the fine-grained vorticity $\omega(\mathbf{r}, t)$ exactly because they develop small-scale filaments that ultimately lead to numerical instabilities. In addition, we are generally not interested in the small scales but only in the largest scales[§]. Indeed, the observations

§ The same problem arises in the kinetic theory of gases. We are not interested to know the position and the velocity of all the molecules of the gas but only some averaged quantities like the temperature or the pressure. The “molecular chaos” at small scales (due to collisions) leads to some mixing, and this is precisely why the velocities achieve the universal Maxwell-Boltzmann distribution (most probable state). Similarly, the statistical theory of 2D turbulence attempts to interpret the coherent structures (vortices and jets) in terms of appropriate Boltzmann distributions corresponding to the most probable or most mixed state of the 2D Euler equation.

are always realized with a finite resolution. It is therefore desirable to have an equation for the coarse-grained vorticity $\bar{\omega}(\mathbf{r}, t)$ that parametrizes at best the small scales and that describes the evolution of the large scales only. Usual parametrizations introduce a turbulent viscosity in the Euler equations in order to smooth out the small-scale filaments and prevent numerical instabilities. However, this artificial viscosity breaks the conservation of energy. It is therefore important to consider more general parametrizations of 2D turbulence that smooth out the small scales while conserving the energy.

In relation to the statistical theory of the 2D Euler equation, some relaxation equations have been proposed by Robert and Sommeria (1992) based on a phenomenological Maximum Entropy Production Principle (MEPP). These equations are constructed so as to increase the mixing entropy (H -theorem) while conserving all the invariants of the inviscid dynamics (the energy and the Casimirs) and to relax towards the maximum entropy state. These equations provide a thermodynamical parametrization of 2D turbulence. In this approach, the energy is conserved globally thanks to a uniform Lagrange multiplier $\beta(t)$ which has the interpretation of a global inverse temperature.

In this paper, we propose to apply the MEPP by enforcing a *local* conservation of energy. This leads to an equation for the vorticity distribution that conserves all the Casimirs, the energy, and that increases monotonically the mixing entropy. However, this equation does not generally converge towards the statistical equilibrium state of the 2D Euler equation. The equation for the coarse-grained vorticity dissipates monotonically all the generalized enstrophies and the vorticity current vanishes for *any* steady state of the 2D Euler equation^{||}. The steady state that is selected is non universal. It is determined by the dynamics in a non-trivial manner (we have to solve a dynamical equation). Interestingly, the equation for the coarse-grained vorticity obtained from the MEPP turns out to coincide with a particular case of equations obtained by Sadourny and Basdevant (1981) with the anticipated vorticity method. Our approach therefore reveals an interesting connection between the anticipated vorticity method and the MEPP when the energy conservation is treated locally.

2. Statistical theory of the 2D Euler equation

Two-dimensional incompressible and inviscid flows are described by the 2D Euler-Poisson system

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = 0, \quad \omega = -\Delta \psi, \quad (1)$$

where ω is the vorticity and ψ the streamfunction. They are related to the velocity field \mathbf{u} by $\nabla \times \mathbf{u} = \omega \mathbf{z}$ and $\mathbf{u} = -\mathbf{z} \times \nabla \psi$, where \mathbf{z} is a unit vector normal to the flow. Starting from a generically unsteady or unstable initial condition, the 2D Euler-Poisson system

^{||} We do not know whether this property is a drawback of the derived equation or if it can account for (observed) situations where the QSS is different from the statistical equilibrium state.

is known to undergo a complicated mixing process. The vorticity $\omega(\mathbf{r}, t)$ develops a filamentation at smaller and smaller scales and never reaches a steady state. However, if we locally average over these filaments, the resulting coarse-grained vorticity $\bar{\omega}(\mathbf{r}, t)$ is expected to reach a quasistationary state (QSS). This is known as weak convergence in mathematics. This QSS, which is a particular steady state of the 2D Euler equation, usually has the form of a large-scale vortex or a jet. A nice illustration of this mixing process is given by Sommeria et al. (1991) in connection to the nonlinear development of the Kelvin-Helmholtz instability in a shear layer.

In order to predict the structure of these QSSs as a function of the initial condition, Miller (1990) and Robert and Sommeria (1991) have proposed a statistical theory of the 2D Euler equation. The key idea is to replace the deterministic description of the flow $\omega(\mathbf{r}, t)$ by a probabilistic description where $\rho(\mathbf{r}, \sigma, t)$ gives the probability density of finding the vorticity level $\omega = \sigma$ in \mathbf{r} at time t . It satisfies the normalization condition $\int \rho d\sigma = 1$. The observed (coarse-grained) vorticity field is then expressed as $\bar{\omega}(\mathbf{r}, t) = \int \rho \sigma d\sigma$.

To apply the statistical theory, we first have to specify the constraints. The 2D Euler equation conserves the energy

$$E = \frac{1}{2} \int \bar{\omega} \psi d\mathbf{r} = \frac{1}{2} \int \rho \sigma \psi d\mathbf{r} d\sigma \quad (2)$$

and the fine-grained vorticity distribution

$$\gamma(\sigma) = \int \rho(\mathbf{r}, \sigma) d\mathbf{r}, \quad (3)$$

where $\gamma(\sigma)$ is the total area occupied by the vorticity level σ . This is equivalent to the conservation of the Casimirs $I_h = \int \bar{h}(\bar{\omega}) d\mathbf{r}$ where h is an arbitrary function of the vorticity.

The basic object of the statistical theory is the mixing entropy

$$S[\rho] = - \int \rho(\mathbf{r}, \sigma) \ln \rho(\mathbf{r}, \sigma) d\mathbf{r} d\sigma \quad (4)$$

which counts the number of microstates corresponding to a given macrostate (Robert and Sommeria 1991, Chavanis 2002). The statistical equilibrium state of the 2D Euler equation, which corresponds to the most probable state (i.e. the macrostate that is the most represented at the microscopic level), is obtained by maximizing the mixing entropy (4) while respecting the normalization condition $\int \rho d\sigma = 1$ and conserving all the inviscid invariants of the 2D Euler equation. If the evolution is ergodic, or at least if mixing is efficient enough, the system will evolve towards the statistical equilibrium state of the 2D Euler equation. It is determined by the maximization problem

$$\max_{\rho} \{ S[\rho] \mid E[\bar{\omega}] = E, \int \rho(\mathbf{r}, \sigma) d\mathbf{r} = \gamma(\sigma), \int \rho d\sigma = 1 \}. \quad (5)$$

The critical points of the mixing entropy at fixed E , $\gamma(\sigma)$ and normalization are obtained from the variational principle

$$\delta S - \beta \delta E - \int \alpha(\sigma) \delta \gamma(\sigma) d\sigma - \int \zeta(\mathbf{r}) \delta \left(\int \rho d\sigma \right) d\mathbf{r} = 0, \quad (6)$$

where β (inverse temperature), $\gamma(\sigma)$ (chemical potential) and $\zeta(\mathbf{r})$ are appropriate Lagrange multipliers. This leads to the equilibrium distribution

$$\rho(\mathbf{r}, \sigma) = \frac{1}{Z(\psi(\mathbf{r}))} g(\sigma) e^{-\beta\sigma\psi(\mathbf{r})}, \quad (7)$$

where $Z(\psi) = \int g(\sigma) e^{-\beta\sigma\psi} d\sigma$ is the normalization factor. The coarse-grained vorticity is then given by

$$\bar{\omega} = \frac{\int g(\sigma) \sigma e^{-\beta\sigma\psi} d\sigma}{\int g(\sigma) e^{-\beta\sigma\psi} d\sigma} = -\frac{1}{\beta} \frac{d \ln Z}{d\psi} = f_{\beta,g}(\psi). \quad (8)$$

Differentiating Eq. (8) with respect to ψ , it is easy to show that the local centered variance of the vorticity distribution

$$\omega_2 \equiv \overline{\omega^2} - \bar{\omega}^2 = \int \rho(\sigma - \bar{\omega})^2 d\mathbf{r} \quad (9)$$

is given by

$$\omega_2 = -\frac{1}{\beta} \bar{\omega}'(\psi) = \frac{1}{\beta^2} \frac{d^2 \ln Z}{d\psi^2}. \quad (10)$$

This relation is reminiscent of the fluctuation-dissipation theorem in statistical mechanics. Since $\bar{\omega} = \bar{\omega}(\psi)$, the statistical theory predicts that the coarse-grained vorticity $\bar{\omega}(\mathbf{r})$ is a stationary solution of the 2D Euler equation. On the other hand, since $\bar{\omega}'(\psi) = -\beta\omega_2(\psi)$ with $\omega_2 \geq 0$, the $\bar{\omega} - \psi$ relationship is a monotonic function that is increasing at negative temperatures $\beta < 0$ and decreasing at positive temperatures $\beta > 0$. Therefore, the statistical theory predicts that the QSS (assumed to be the most mixed state) is characterized by a monotonic $\bar{\omega}(\psi)$ relationship. This $\bar{\omega} - \psi$ relationship can take different shapes depending on the initial condition. Substituting Eq. (8) in the Poisson equation (1-b), the statistical equilibrium state is obtained by solving the differential equation

$$-\Delta\psi = f_{\beta,g}(\psi) \quad (11)$$

with adequate boundary conditions, and relating the Lagrange multipliers β and $g(\sigma)$ to the constraints E and $\gamma(\sigma)$. We also have to make sure that the distribution (7) is a (local) maximum of entropy, not a minimum or a saddle point.

3. Inviscid selective decay and generalized enstrophies

The moments of the fine-grained vorticity $\Gamma_n^{f.g.} = \int \bar{\omega}^n d\mathbf{r} = \int \rho \sigma^n d\mathbf{r} d\sigma$ are conserved since they are particular Casimirs. The first moment is the circulation $\Gamma = \int \bar{\omega} d\mathbf{r} = \int \rho \sigma d\mathbf{r} d\sigma$ and the second moment is the fine-grained enstrophy $\Gamma_2^{f.g.} = \int \bar{\omega}^2 d\mathbf{r} = \int \rho \sigma^2 d\mathbf{r} d\sigma$. For $n > 1$ the moments of the coarse-grained vorticity $\Gamma_{n>1}^{c.g.} = \int \bar{\omega}^n d\mathbf{r}$ are *not* conserved since $\bar{\omega}^n \neq \bar{\omega}^n$ (part of the coarse-grained moments goes into fine-grained fluctuations). For example, using the Schwarz inequality, we find that $\Gamma_2^{c.g.} = \int \bar{\omega}^2 d\mathbf{r} \leq \Gamma_2^{f.g.} = \int \bar{\omega}^2 d\mathbf{r}$. Therefore, the enstrophy calculated with the coarse-grained vorticity is always smaller than its initial value while the enstrophy calculated with the fine-grained vorticity is conserved. Therefore, the notion of coarse-graining

explains how we can have a decrease of enstrophy in a purely inviscid theory. By contrast, the circulation and the energy calculated with the coarse-grained vorticity are conserved. For that reason, the energy and the circulation are called robust invariants while the moments of the vorticity of order $n > 1$ are called fragile invariants. These results may be viewed as a form of inviscid selective decay due to coarse-graining (not to viscosity).

We introduce a family of functionals of the coarse-grained vorticity of the form

$$S[\bar{\omega}] = - \int C(\bar{\omega}) d\mathbf{r}, \quad (12)$$

where C is any convex function (i.e. $C'' > 0$). It can be shown that these functionals increase with time (due to mixing) in the sense that $S(t) \geq S(0)$ for any $t > 0$. For that reason they are sometimes called “generalized H -functions” (Tremaine et al. 1986, Appendix A of Chavanis 2006) or “generalized entropies” (Chavanis 2003). However, (i) a monotonic increase of $S(t)$ (i.e. an H -theorem) and (ii) the relaxation of the system towards a maximum of one of these functionals S are *not* implied by this theorem (although they may be expected in generic situations). We note that the neg-*enstrophy* (the opposite of the enstrophy $\Gamma_2^{c.g.} = \int \bar{\omega}^2 d\mathbf{r}$) is a particular case of such functionals. For that reason, the functionals $-S$ are sometimes called “generalized enstrophies”.

4. The Maximum Entropy Production Principle with a global conservation of energy

Let us decompose the vorticity ω and the velocity \mathbf{u} into a mean and a fluctuating part, namely $\omega = \bar{\omega} + \tilde{\omega}$, $\mathbf{u} = \bar{\mathbf{u}} + \tilde{\mathbf{u}}$. Taking the local average of the Euler equation (1-a), we get

$$\frac{\partial \bar{\omega}}{\partial t} + \nabla \cdot (\bar{\omega} \bar{\mathbf{u}}) = -\nabla \cdot \mathbf{J}_\omega, \quad (13)$$

where the vorticity current $\mathbf{J}_\omega = \overline{\tilde{\omega} \tilde{\mathbf{u}}}$ represents the correlations of the fine-grained fluctuations. Eq. (13) can be viewed as a local conservation law for the circulation $\Gamma = \int \bar{\omega} d\mathbf{r}$. In order to conserve all the Casimirs, we need to consider not only the locally averaged vorticity field $\bar{\omega}(\mathbf{r}, t)$ but the whole probability distribution $\rho(\mathbf{r}, \sigma, t)$ now evolving with time t . The conservation of the global vorticity distribution $\gamma(\sigma) = \int \rho d\mathbf{r}$ can be written in the local form as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{\mathbf{u}}) = -\nabla \cdot \mathbf{J}, \quad (14)$$

where $\mathbf{J}(\mathbf{r}, \sigma, t)$ is the (unknown) current associated with the vorticity level σ . Integrating Eq. (14) over all the vorticity levels σ , we find the constraint $\int \mathbf{J}(\mathbf{r}, \sigma, t) d\sigma = \mathbf{0}$. This accounts for the conservation of the normalization condition $\int \rho d\sigma = 1$. Multiplying Eq. (14) by σ and integrating over all the vorticity levels, we get $\int \mathbf{J}(\mathbf{r}, \sigma, t) \sigma d\sigma = \mathbf{J}_\omega$.

We can express the time variation of energy in terms of \mathbf{J}_ω , using Eqs. (2) and (13). This leads to the constraint

$$\dot{E} = \int \mathbf{J}_\omega \cdot \nabla \psi \, d\mathbf{r} = 0. \quad (15)$$

Using Eqs. (4) and (14), we similarly express the rate of entropy production as

$$\dot{S} = - \int \mathbf{J} \cdot \frac{\nabla \rho}{\rho} \, d\mathbf{r} d\sigma. \quad (16)$$

The Maximum Entropy Production Principle (MEPP) consists in choosing the current \mathbf{J} which maximizes the rate of entropy production \dot{S} respecting the constraints $\dot{E} = 0$, $\int \mathbf{J} \, d\sigma = \mathbf{0}$, and $\int \frac{J^2}{2\rho} \, d\sigma \leq C(\mathbf{r}, t)$. The last constraint expresses a bound (unknown) on the value of the diffusion current. Convexity arguments justify that this bound is always reached so that the inequality can be replaced by an equality. The corresponding condition on first variations can be written at each time t :

$$\delta \dot{S} - \beta(t) \delta \dot{E} - \int \zeta(\mathbf{r}, t) \delta \left(\int \mathbf{J} \, d\sigma \right) d\mathbf{r} - \int \frac{1}{D(\mathbf{r}, t)} \delta \left(\int \frac{\mathbf{J}^2}{2\rho} d\sigma \right) d\mathbf{r} = 0 \quad (17)$$

and leads to a current of the form

$$\mathbf{J} = -D(\mathbf{r}, t) [\nabla \rho + \beta(t) \rho (\sigma - \bar{\omega}) \nabla \psi]. \quad (18)$$

The Lagrange multiplier $\zeta(\mathbf{r}, t)$ has been eliminated, using the condition $\int \mathbf{J} \, d\sigma = \mathbf{0}$ of local normalization conservation. The vorticity current is

$$\mathbf{J}_\omega = -D(\mathbf{r}, t) [\nabla \bar{\omega} + \beta(t) \omega_2 \nabla \psi]. \quad (19)$$

The thermodynamical parametrization proposed by Robert & Sommeria (1992) can therefore be written as

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = \nabla \cdot \left\{ D(\mathbf{r}, t) [\nabla \rho + \beta(t) \rho (\sigma - \bar{\omega}) \nabla \psi] \right\}. \quad (20)$$

The equation for the coarse-grained vorticity is

$$\frac{\partial \bar{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \bar{\omega} = \nabla \cdot \left\{ D(\mathbf{r}, t) [\nabla \bar{\omega} + \beta(t) \omega_2 \nabla \psi] \right\}. \quad (21)$$

We note that this equation is not closed since it depends on the local centered enstrophy ω_2 . We therefore have to solve Eq. (20) for all the levels or write an infinite hierarchy of equations for the moments $\bar{\omega}^k$ (Robert and Rosier 1997). The time evolution of the Lagrange multiplier $\beta(t)$ is determined by introducing Eq. (19) in the constraint (15). This yields

$$\beta(t) = - \frac{\int D \nabla \bar{\omega} \cdot \nabla \psi \, d\mathbf{r}}{\int D \omega_2 (\nabla \psi)^2 \, d\mathbf{r}}. \quad (22)$$

The mixing entropy (4) satisfies an H -theorem provided that $D \geq 0$. Indeed, using the expression (18) of the current, the entropy production (16) can be rewritten as

$$\dot{S} = \int \frac{\mathbf{J}^2}{D\rho} \, d\mathbf{r} d\sigma + \int \frac{\mathbf{J}}{\rho} \cdot \{ \beta(t) \rho (\sigma - \bar{\omega}) \nabla \psi \} \, d\mathbf{r} d\sigma. \quad (23)$$

Integrating over the vorticity levels in the second term, and using the conservation of energy (15), we get

$$\dot{S} = \int \frac{\mathbf{J}^2}{D\rho} d\mathbf{r}d\sigma + \beta(t) \int \mathbf{J}_\omega \cdot \nabla\psi d\mathbf{r} = \int \frac{\mathbf{J}^2}{D\rho} d\mathbf{r}d\sigma \geq 0. \quad (24)$$

A stationary solution of Eq. (20) satisfies $\dot{S} = 0$ implying $\mathbf{J} = \mathbf{0}$. Using Eq. (18), we obtain $\nabla \ln \rho + \beta(\sigma - \bar{\omega})\nabla\psi = \mathbf{0}$. For any reference vorticity level σ_0 , it writes $\nabla \ln \rho_0 + \beta(\sigma_0 - \bar{\omega})\nabla\psi = \mathbf{0}$. Subtracting the foregoing equations, we obtain $\nabla \ln(\rho/\rho_0) + \beta(\sigma - \sigma_0)\nabla\psi = \mathbf{0}$, which is immediately integrated into Eq. (7) where $Z^{-1}(\mathbf{r}) \equiv \rho_0(\mathbf{r})e^{\beta\sigma_0\psi(\mathbf{r})}$ and $g(\sigma) \equiv e^{A(\sigma)}$, $A(\sigma)$ being a constant of integration. Therefore, the mixing entropy (4) increases monotonically until the distribution (7) is reached, with $\beta = \lim_{t \rightarrow \infty} \beta(t)$. It can be shown that a stationary solution of the relaxation equation (20) is linearly stable if, and only if, it is an entropy *maximum* at fixed energy and Casimirs. Therefore, this numerical algorithm selects the maxima (and not the minima or the saddle points) among all the critical points of entropy. When several entropy maxima subsist for the same values of the constraints, the choice of equilibrium depends on a complicated notion of “basin of attraction” and not simply whether the solution is a local or a global entropy maximum.

In summary, Eq. (20) conserves the energy (2), the Casimirs (3), and increases monotonically the mixing entropy (4) (*H*-theorem). For $t \rightarrow +\infty$, the solution converges towards the maximum entropy state (7). The generalized enstrophies (12) are not conserved but it does not seem possible to prove whether they decay monotonically or not (actually, there is no fundamental reason why they should decay monotonically in an inviscid theory).

The vorticity current in the relaxation equation (21) is the sum of two terms. A term $\mathbf{J}_{diff} = -D\nabla\bar{\omega}$ leading to a pure diffusion with a turbulent viscosity D and an additional term $\mathbf{J}_{drift} = -\beta D\omega_2\nabla\psi$ interpreted as a *drift*. The drift coefficient (mobility) is given by a sort of Einstein relation¶. The relaxation equation (21) may be interpreted as a nonlinear Fokker-Planck equation. It shares some analogies with the Fokker-Planck equation obtained in the kinetic theory of point vortices in the thermal bath approximation (Chavanis 2001, 2002) although the physics of the problem is fundamentally different. On the other hand, a kinetic equation for the coarse-grained vorticity has been derived from a quasilinear theory of the 2D Euler equation (Chavanis 2000, 2002), and some connections with the relaxation equations issued from the MEPP have been mentioned.

Usual parameterizations of 2D turbulence include a single turbulent viscosity. In a sense, they correspond to the infinite temperature limit ($\beta = 0$) of the relaxation equation (21) where the drift vanishes. However, these equations without the drift term do not conserve the energy. The drift term is therefore necessary to restore this property.

¶ There exist numerous analogies between the kinetic theories of 2D vortices and stellar systems. In these analogies, the drift of the vortices is the counterpart of the dynamical friction experienced by a star (Chavanis 2002).

5. The Maximum Entropy Production Principle with a local conservation of energy

In the previous section, the energy is conserved globally thanks to a uniform Lagrange multiplier $\beta(t)$ interpreted as a global inverse temperature. In this section, we propose to apply the MEPP by imposing a local conservation of energy $\mathbf{J}_\omega \cdot \nabla \psi = 0$. In that case, the variational problem (17) is replaced by

$$\begin{aligned} \delta \dot{S} - \int \beta(\mathbf{r}, t) \delta(\mathbf{J}_\omega \cdot \nabla \psi) d\mathbf{r} - \int \zeta(\mathbf{r}, t) \delta \left(\int \mathbf{J} d\sigma \right) d\mathbf{r} \\ - \int \frac{1}{D(\mathbf{r}, t)} \delta \left(\int \frac{\mathbf{J}^2}{2\rho} d\sigma \right) d\mathbf{r} = 0. \end{aligned} \quad (25)$$

It leads to an optimal current of the form

$$\mathbf{J} = -D(\mathbf{r}, t) [\nabla \rho + \beta(\mathbf{r}, t) \rho (\sigma - \bar{\omega}) \nabla \psi]. \quad (26)$$

The vorticity current is

$$\mathbf{J}_\omega = -D(\mathbf{r}, t) [\nabla \bar{\omega} + \beta(\mathbf{r}, t) \omega_2 \nabla \psi]. \quad (27)$$

The evolution of the Lagrange multiplier $\beta(\mathbf{r}, t)$ is determined by introducing Eq. (27) in the local energy constraint $\mathbf{J}_\omega \cdot \nabla \psi = 0$. This yields $\nabla \psi \cdot \nabla \bar{\omega} + \beta \omega_2 (\nabla \psi)^2 = 0$ implying

$$\beta(\mathbf{r}, t) = -\frac{\nabla \psi \cdot \nabla \bar{\omega}}{\omega_2 (\nabla \psi)^2}. \quad (28)$$

Substituting these expressions in Eqs. (13) and (14), we obtain the parametrization

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = \nabla \cdot \left\{ D(\mathbf{r}, t) \left[\nabla \rho - \rho (\sigma - \bar{\omega}) \frac{\nabla \psi \cdot \nabla \bar{\omega}}{\omega_2 (\nabla \psi)^2} \nabla \psi \right] \right\} \quad (29)$$

and

$$\frac{\partial \bar{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \bar{\omega} = \nabla \cdot \left\{ D(\mathbf{r}, t) \left[\nabla \bar{\omega} - \frac{\nabla \psi \cdot \nabla \bar{\omega}}{(\nabla \psi)^2} \nabla \psi \right] \right\}. \quad (30)$$

Since the diffusion coefficient is unspecified, we can take $D \propto (\nabla \psi)^2$ to avoid dividing by zero when $\nabla \psi = \mathbf{0}$.

These equations conserve the normalization, the total surface of each vorticity level, and the energy. They also increase the mixing entropy (4) monotonically provided that $D \geq 0$. Indeed, using the expression (26) of the current, the entropy production (16) can be rewritten as

$$\dot{S} = \int \frac{\mathbf{J}^2}{D\rho} d\mathbf{r} d\sigma + \int \frac{\mathbf{J}}{\rho} \cdot [\beta(\mathbf{r}, t) \rho (\sigma - \bar{\omega}) \nabla \psi] d\mathbf{r} d\sigma. \quad (31)$$

Integrating over the vorticity levels, and using the local conservation of energy, we get

$$\dot{S} = \int \frac{\mathbf{J}^2}{D\rho} d\mathbf{r} d\sigma + \int \beta(\mathbf{r}, t) \mathbf{J}_\omega \cdot \nabla \psi d\mathbf{r} = \int \frac{\mathbf{J}^2}{D\rho} d\mathbf{r} d\sigma \geq 0. \quad (32)$$

It is also possible to prove that Eq. (30) for the coarse-grained vorticity dissipates all the generalized enstrophies (12) monotonically provided that $D \geq 0$. The rate of dissipation of the generalized enstrophies is the opposite of

$$\dot{S} = - \int C''(\bar{\omega}) \mathbf{J}_\omega \cdot \nabla \bar{\omega} d\mathbf{r}. \quad (33)$$

Using the expression (27) of the vorticity current and the local conservation of energy, we get

$$\dot{S} = \int C''(\omega) \frac{\mathbf{J}_\omega^2}{D} d\mathbf{r} + \int C'''(\bar{\omega}) \beta \omega_2 \mathbf{J}_\omega \cdot \nabla \psi d\mathbf{r} = \int C''(\omega) \frac{\mathbf{J}_\omega^2}{D} d\mathbf{r} \geq 0. \quad (34)$$

In summary, Eq. (29) conserves the energy (2), the Casimirs (3), and increases monotonically the mixing entropy (4) (*H*-theorem). We emphasize, however, that this equation does *not* relax towards the maximum entropy state in general. Indeed, Eq. (30) for the coarse-grained vorticity dissipates all the generalized enstrophies (12) monotonically. Furthermore, the vorticity current vanishes for *any* steady state of the 2D Euler equation such that $\bar{\omega} = f(\psi)$. Therefore, the system is expected to reach a steady state of the 2D Euler equation but its precise form cannot be determined *a priori*. It is not universal. It depends on the dynamics and we have to solve Eq. (30).

We note the remarkable fact that the equation for the coarse-grained vorticity (30) is *closed*. This is not the case in the parametrization (21) where it depends on the local enstrophy $\omega_2(\mathbf{r}, t)$. This is a great practical advantage of the present parametrization since we do not have to solve the equation for the vorticity distribution (29), or consider an infinite hierarchy of moments equations, to obtain the evolution of the coarse-grained vorticity (which is the quantity of main interest)⁺. This “miracle” only occurs for the coarse-grained vorticity. The evolution of the higher order moments $\bar{\omega}^k(\mathbf{r}, t)$ are given by an infinite hierarchy of equations obtained from Eq. (29).

6. The anticipated vorticity method

In the presence of a very small viscosity, the energy and the circulation are almost conserved while the enstrophy decays monotonically (actually, all the generalized enstrophies decay monotonically, see Appendix A of Chavanis 2006). This property of selective decay has led to the minimum enstrophy principle*. It also suggests to develop a parametrization that conserves the energy and the circulation while dissipating monotonically the enstrophy (or the generalized enstrophies). These considerations have lead to the parametrization of Sadourny and Basdevant (1981) based on the anticipated vorticity method.

Following Sadourny and Basdevant (1981), we determine the vorticity current \mathbf{J}_ω in order to conserve locally the energy and decrease monotonically all the generalized enstrophies (Sadourny and Basdevant only consider the dissipation of enstrophy but we show below that, actually, all the generalized enstrophies decay). We assume that the energy is conserved locally so that $\mathbf{J}_\omega \cdot \nabla \psi = 0$. This implies that the vorticity

⁺ The fact that the results depend on the detailed vorticity distribution $\rho(\mathbf{r}, \sigma, t)$, as implied by the statistical theory of 2D turbulence, is the main practical difficulty to implement the parametrization (20). In realistic applications, it is difficult to determine what are the levels to consider.

* This “principle” is only phenomenological. Indeed, the conservation of energy and circulation, and the monotonic decay of enstrophy, do not guarantee that the system will necessarily reach a minimum enstrophy state at fixed energy and circulation.

current must be parallel to the velocity, i.e. $\mathbf{J}_\omega = -\lambda(\mathbf{r}, t)\mathbf{u}$ where $\lambda(\mathbf{r}, t)$ is an arbitrary function. Substituting this relation in Eq. (33) we get $\dot{S} = \int C''(\bar{\omega})\lambda(\mathbf{r}, t)\mathbf{u} \cdot \nabla \bar{\omega} d\mathbf{r}$. If we take $\lambda(\mathbf{r}, t) = K(\mathbf{r}, t)\mathbf{u} \cdot \nabla \bar{\omega}$ with $K \geq 0$, we obtain $\dot{S} \geq 0$. Finally, it is relevant to write $K = D/u^2$ where $D(\mathbf{r}, t) \geq 0$ has the dimension of a diffusion coefficient. Therefore $\mathbf{J}_\omega = -D(\mathbf{u} \cdot \nabla \bar{\omega})\mathbf{u}/u^2$. Substituting this expression in Eq. (13), we obtain

$$\frac{\partial \bar{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \bar{\omega} = \nabla \cdot \left[D \frac{(\mathbf{u} \cdot \nabla \bar{\omega})\mathbf{u}}{u^2} \right]. \quad (35)$$

This equation can also be written as

$$\frac{\partial \bar{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \bar{\omega} = \nabla \cdot \left(D \frac{\mathbf{u} \otimes \mathbf{u}}{u^2} \nabla \bar{\omega} \right). \quad (36)$$

Since D is unspecified, we can take $D = Ku^2$ in order to avoid dividing by zero when $u = 0$. Eq. (35) is a particular case of the parametrization proposed by Sadourny and Basdevant (1981). This equation conserves locally the energy and dissipates monotonically all the generalized enstrophies:

$$\dot{S} = \int C''(\bar{\omega}) \frac{D}{u^2} (\mathbf{u} \cdot \nabla \bar{\omega})^2 d\mathbf{r} \geq 0. \quad (37)$$

The diffusion current vanishes when a steady state of the 2D Euler equation is reached. Indeed, $\mathbf{J}_\omega = 0$ when $\mathbf{u} \cdot \nabla \bar{\omega} = 0$ which is equivalent to $\bar{\omega} = f(\psi)$. However, it does not seem possible to predict that steady state *a priori*. Its selection depends on the dynamics in a non-trivial manner and we have to solve Eq. (35).

Finally, we show that Eq. (35) is equivalent to Eq. (30) derived from the MEPP. Combining the identity of vector analysis

$$\mathbf{u} \times (\nabla \bar{\omega} \times \mathbf{u}) = u^2 \nabla \bar{\omega} - (\mathbf{u} \cdot \nabla \bar{\omega})\mathbf{u} \quad (38)$$

with the relation

$$\nabla \bar{\omega} \times \mathbf{u} = \nabla \bar{\omega} \times (-\mathbf{z} \times \nabla \psi) = -(\nabla \bar{\omega} \cdot \nabla \psi)\mathbf{z} \quad (39)$$

leading to

$$\mathbf{u} \times (\nabla \bar{\omega} \times \mathbf{u}) = (\nabla \bar{\omega} \cdot \nabla \psi) \nabla \psi, \quad (40)$$

we find that

$$\frac{(\mathbf{u} \cdot \nabla \bar{\omega})\mathbf{u}}{u^2} = \nabla \bar{\omega} - \frac{(\nabla \bar{\omega} \cdot \nabla \psi) \nabla \psi}{(\nabla \psi)^2}. \quad (41)$$

Therefore, Eq. (35) is the same as Eq. (30).

7. Selective decay principle

As we have seen, Eq. (35) is equivalent to Eq. (30). However, the form (30) of this equation, which does not seem to have been noticed before, has a very interesting structure. The right hand side of Eq. (30) is the sum of two terms. A diffusion term and a drift term. Usual parameterizations only include a diffusion term. However, this diffusion term alone dissipates the energy which is a bad feature of these

parametrizations. We show below that the drift term acts precisely in a way to restore the conservation of energy. Indeed, we can write $\dot{E} = \dot{E}_{diff} + \dot{E}_{drift}$ with

$$\dot{E}_{diff} = \int \mathbf{J}_{diff} \cdot \nabla \psi \, d\mathbf{r} = - \int D \nabla \bar{\omega} \cdot \nabla \psi \, d\mathbf{r} = -D \int \bar{\omega}^2 \, d\mathbf{r} \leq 0, \quad (42)$$

$$\dot{E}_{drift} = \int \mathbf{J}_{drift} \cdot \nabla \psi \, d\mathbf{r} = \int D \frac{\nabla \psi \cdot \nabla \bar{\omega}}{(\nabla \psi)^2} \nabla \psi \cdot \nabla \psi \, d\mathbf{r} = D \int \bar{\omega}^2 \, d\mathbf{r} \geq 0. \quad (43)$$

In order to obtain the last integral, we have assumed that D is constant and used an integration by parts (but this last step is not necessary for the proof). We see that $\dot{E}_{drift} = -\dot{E}_{diff} \geq 0$. The diffusion term dissipates the energy while the drift term increases it. As a whole, the energy is conserved: $\dot{E} = 0$.

A pure diffusion term dissipates the generalized enstrophies monotonically (see Appendix A of Chavanis 2006). We show below that this property persists in the presence of the drift term. Indeed, we can write $\dot{S} = \dot{S}_{diff} + \dot{S}_{drift}$ with

$$\dot{S}_{diff} = - \int C''(\bar{\omega}) \mathbf{J}_{diff} \cdot \nabla \bar{\omega} \, d\mathbf{r} = \int D C''(\bar{\omega}) (\nabla \bar{\omega})^2 \, d\mathbf{r} \geq 0, \quad (44)$$

$$\dot{S}_{drift} = - \int C''(\bar{\omega}) \mathbf{J}_{drift} \cdot \nabla \bar{\omega} \, d\mathbf{r} = - \int D C''(\bar{\omega}) \frac{(\nabla \psi \cdot \nabla \bar{\omega})^2}{(\nabla \psi)^2} \, d\mathbf{r} \leq 0. \quad (45)$$

The diffusion term dissipates the generalized enstrophies while the drift term increases them. As a whole, the generalized enstrophies decay monotonically: $\dot{S} \geq 0$.

These properties are strikingly consistent with the phenomenology of 2D turbulence. With only the diffusion term, we have a direct cascade of enstrophy and a spurious direct cascade of energy (for a “large” turbulent viscosity). With the diffusion term and the drift term, we have a direct cascade of enstrophy and an inverse cascade of energy. Eq. (30) is therefore consistent with the phenomenological selective decay principle. It dissipates the generalized enstrophies while conserving the energy.

Remark: We can obtain similar relations for the parametrization (20) associated with the entropy (4). We first have $\dot{E}_{drift} = -\dot{E}_{diff} = - \int D \nabla \bar{\omega} \cdot \nabla \psi \, d\mathbf{r} \geq 0$ leading to $\dot{E} = 0$. We also find that $\dot{S}_{diff} = \int D [(\nabla \rho)^2 / \rho] \, d\mathbf{r} \geq 0$ and $\dot{S}_{drift} = \beta(t) \dot{E}_{drift}$. If $\beta(t) \leq 0$, then $\dot{S}_{drift} \leq 0$. In any case, $\dot{S} \geq 0$.

8. Differences with other equations

Eq. (30) is different from the relaxation equation

$$\frac{\partial \bar{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \bar{\omega} = \nabla \cdot \left[D(\mathbf{r}, t) \left(\nabla \bar{\omega} + \frac{\beta(t)}{C''(\bar{\omega})} \nabla \psi \right) \right], \quad (46)$$

$$\beta(t) = - \frac{\int D \nabla \bar{\omega} \cdot \nabla \psi \, d\mathbf{r}}{\int D \frac{(\nabla \psi)^2}{C''(\bar{\omega})} \, d\mathbf{r}} \quad (47)$$

derived by Chavanis (2003) from a generalized MEPP in $\bar{\omega}$ -space. This equation increases monotonically a *particular* generalized entropy S , specified by the convex function $C(\bar{\omega})$, while conserving energy E and circulation Γ . It relaxes towards a (local)

maximum of S at fixed Γ and E . We note that Eqs. (46) and (47) may be obtained from Eqs. (21) and (22) by making the *ansatz* $\omega_2(\mathbf{r}, t) = 1/C''[\bar{\omega}(\mathbf{r}, t)]$. We also note that Eq. (30) can be obtained from Eqs. (46) and (47) if the global conservation of energy is replaced by a local conservation of energy, i.e. if we suppress the integrals in Eq. (47). This shows that the decay of the generalized enstrophies in Eq. (30) is optimal.

Eq. (30) is the “opposite” of the equation

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = -\alpha \{\omega, \{\omega, \psi\}\} \quad (48)$$

proposed by Vallis et al. (1989). For $\alpha < 0$, this equation dissipates the energy monotonically while conserving all the Casimirs. It relaxes towards the minimum of energy under isovortical perturbations.

We refer to Chavanis (2009) for a more detailed discussion of the variational problems of 2D turbulence and the corresponding relaxation equations.

9. Discussion

Robert and Sommeria (1992) have used a MEPP with a global conservation of energy and derived the parametrization (20). It is possible to take into account the conservation of angular momentum $L = \int \omega r^2 d\mathbf{r}$ and linear impulse $\mathbf{P} = -\mathbf{z} \times \int \omega \mathbf{r} d\mathbf{r}$ in their parametrization by introducing appropriate Lagrange multipliers $\Omega(t)$ and $\mathbf{U}(t)$ in the variational principle (17). In that case, the stream function $\psi(\mathbf{r}, t)$ is replaced by the relative stream function $\psi_{eff}(\mathbf{r}, t) = \psi(\mathbf{r}, t) + \frac{\Omega(t)}{2} r^2 - \mathbf{U}_\perp(t) \cdot \mathbf{r}$ (Chavanis et al. 1996). This parametrization works well to describe the organization of a 2D turbulent flow into a *single* coherent structure, for example the vortex resulting from the merging of two vortices (Robert and Sommeria 1992, Robert and Rosier 1997). However, this parametrization does not respect the galilean invariance of the 2D Euler equation. In addition, the conservation of energy, angular momentum, and linear impulse are enforced globally thanks to uniform Lagrange multipliers $\beta(t)$, $\Omega(t)$ and $\mathbf{U}(t)$. Not only this procedure is artificial, but it also poses practical problems to describe large-scale flows that organize into several *distinct* coherent structures. Indeed, if we view these individual structures as maximum entropy states, there is no reason why they should all have the same temperature, angular velocity, and linear impulse. In addition, in order to determine $\beta(t)$, $\Omega(t)$ and $\mathbf{U}(t)$ in the parametrization (20) we have to integrate over the whole domain while the physics of the problem should be more local, even though the interaction is long-range. For example, to describe the formation of a large cyclone over a part of the world, it should not be necessary to perform an integral over the whole sphere (the Earth) as implied by Eq. (22). To solve these problems, Chavanis and Sommeria (1997) have proposed a set of relaxation equations that preserve the galilean invariance of the 2D Euler equation and that satisfy the conservation of energy, angular momentum and linear impulse locally thanks to diffusion currents. These equations relax towards individual coherent structures that correspond to statistical equilibrium states

(7) having *different* values of temperature, angular velocity, and linear impulse. They may be interpreted as “maximum entropy bubbles” (Chavanis and Sommeria 1998). For the moment, this parametrization has never been used in practice. One difficulty is that we have to consider a large number of coupled equations for each level σ or an infinite hierarchy of equations for the vorticity moments $\overline{\omega^k}(\mathbf{r}, t)$ (a difficulty inherent to the statistical theory of 2D turbulence). In the present paper, we have considered a different approach. We have used the MEPP with a local conservation of energy \ddagger and derived the parametrization (29)-(30). We have shown that this parametrization is equivalent to a special case (35) of the parametrization of Sadourny and Basdevant (1981) based on the anticipated vorticity method although the equations appear in a different form [compare Eqs. (30) and (35)]. The new form of equation (30) derived in the present paper has a more physical interpretation than the form (35) (see Sec. 7). An advantage of this parametrization over the parameterizations of Robert and Sommeria (1992) and Chavanis and Sommeria (1997) is that it yields a closed equation for the coarse-grained vorticity $\overline{\omega}(\mathbf{r}, t)$ instead of an infinite hierarchy of equations for the vorticity moments $\overline{\omega^k}(\mathbf{r}, t)$. However, it does not respect the conservation of angular momentum and linear impulse, nor the galilean invariance of the 2D Euler equation. Finally, it does not in general relax towards a maximum entropy state unlike the parameterizations of Robert and Sommeria (1992) and Chavanis and Sommeria (1997). Indeed, in the parametrization (30) or (35) the diffusion current vanishes for any steady state of the 2D Euler equation. This may account for non-ergodicity or be a drawback of this parametrization. Finally, the relaxation equation (46) of Chavanis (2003) is somehow intermediate between these different parameterizations since it relaxes towards a special class of steady states of the 2D Euler equations specified by a generalized entropy $S[\overline{\omega}]$. It is straightforward to generalize this equation in order to conserve energy, angular momentum and linear impulse locally by making the *ansatz* $\omega_2(\mathbf{r}, t) = 1/C''[\overline{\omega}(\mathbf{r}, t)]$ in the parametrization of Chavanis and Sommeria (1997). Of course, it would be interesting to compare the efficiency of these different parameterizations through direct numerical simulations.

10. Conclusion

We have shown a connection between the anticipated vorticity method of Sadourny and Basdevant (1981) and the Maximum Entropy Production Principle of Robert and Sommeria (1992) when the conservation of energy is treated locally (instead of globally). This connection is new and is the main result of the paper. More than showing the relation between two known equations, we have derived from the MEPP a new form of equation [see Eq. (30)] that turns out, after some algebraic manipulations, to coincide with a special case of the parametrization of Sadourny and Basdevant (1981) [see Eq. (35)]. The new form of equation (30), which incorporates a diffusion term and a drift

\ddagger The difference with Chavanis and Sommeria (1998) is that we take here the current of energy \mathbf{J}_ϵ equal to zero, i.e. we impose $\mathbf{J}_\omega \cdot \nabla \psi = 0$ instead of $\mathbf{J}_\omega \cdot \nabla \psi = \nabla \cdot \mathbf{J}_\epsilon$ with $\mathbf{J}_\epsilon \neq \mathbf{0}$.

term, has a nice physical interpretation in terms of a selective decay principle. This gives a new light to both the MEPP and the anticipated vorticity method.

Appendix A. The equation for the velocity field

The equation for the coarse-grained velocity field is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p - \mathbf{z} \times \mathbf{J}_\omega, \quad (\text{A.1})$$

where p is the pressure and \mathbf{J}_ω the vorticity current. Taking the curl of Eq. (A.1) and using the identity $\nabla \times (\mathbf{z} \times \mathbf{a}) = (\nabla \cdot \mathbf{a}) \mathbf{z}$ and the definition $\nabla \times \mathbf{u} = \bar{\omega} \mathbf{z}$ we recover Eq. (13). In Sec. 5, we have established that

$$\mathbf{J}_\omega = -D \left(\nabla \bar{\omega} - \frac{\nabla \bar{\omega} \cdot \nabla \psi}{(\nabla \psi)^2} \nabla \psi \right). \quad (\text{A.2})$$

Using the identity of vector analysis $\Delta \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})$ which reduces for a 2D incompressible flow to $\Delta \mathbf{u} = \mathbf{z} \times \nabla \bar{\omega}$ we obtain

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + D \left(\Delta \mathbf{u} + \frac{\nabla \bar{\omega} \cdot \nabla \psi}{(\nabla \psi)^2} \mathbf{u} \right). \quad (\text{A.3})$$

Under this form, the drift is equivalent to a force directed along the velocity \mathbf{u} . It points in the same direction as the velocity when $\nabla \bar{\omega} \cdot \nabla \psi > 0$ which corresponds to negative temperatures. It may therefore be interpreted as an anti-friction, or a forcing, that restores the conservation of energy dissipated by the diffusion term. Using the equivalent expression

$$\mathbf{J}_\omega = -D \frac{(\mathbf{u} \cdot \nabla \bar{\omega}) \mathbf{u}}{u^2} \quad (\text{A.4})$$

of the vorticity current (see Sec. 6), we obtain

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + D \frac{(\mathbf{u} \cdot \nabla \bar{\omega}) \nabla \psi}{u^2}. \quad (\text{A.5})$$

Remark: With the parametrization of Appendix C, the turbulent terms in Eqs. (A.3) and (A.5) are replaced by $D(\Delta \mathbf{u} - \frac{\mathbf{r} \cdot \nabla \bar{\omega}}{r^2} \mathbf{r}_\perp)$ and $-D \frac{(\mathbf{r}_\perp \cdot \nabla \bar{\omega})}{r^2} \mathbf{r}$ respectively.

Appendix B. Application to the shallow-water equations

Chavanis and Sommeria (2002) have developed a statistical theory of the shallow-water (SW) equations and they have derived a set of relaxation equations towards the statistical equilibrium state by using a MEPP. This is a generalization of the parametrization (20). The relaxation equations (46)-(47) have been generalized to the SW equations by Chavanis and Dubrulle (2006). Finally, it is straightforward to generalize the parametrization (29) to the SW equations. This leads to the following set of equations

$$\frac{\partial h}{\partial t} + \nabla \cdot (h \mathbf{u}) = 0, \quad (\text{B.1})$$

$$\frac{\partial \mathbf{u}}{\partial t} + \bar{q} h \mathbf{z} \times \mathbf{u} = -\nabla B - \mathbf{z} \times \mathbf{J}_\omega, \quad (\text{B.2})$$

$$B = gh + \frac{\mathbf{u}^2}{2}, \quad \bar{q} = \frac{\omega + 2\Omega}{h}, \quad (\text{B.3})$$

$$\frac{\partial}{\partial t}(h\bar{q}) + \nabla \cdot (h\bar{q}\mathbf{u}) = -\nabla \cdot \mathbf{J}_\omega, \quad (\text{B.4})$$

$$\mathbf{J}_\omega = -D(\mathbf{r}, t) \left(\nabla \bar{q} - \frac{\mathbf{u}_\perp \cdot \nabla \bar{q}}{u_\perp^2} \mathbf{u}_\perp \right), \quad (\text{B.5})$$

$$\frac{\partial}{\partial t}(h\rho) + \nabla \cdot (h\rho\mathbf{u}) = -\nabla \cdot \mathbf{J}, \quad (\text{B.6})$$

$$\mathbf{J} = -D(\mathbf{r}, t) \left[\nabla \rho - \rho(\sigma - \bar{q}) \frac{\mathbf{u}_\perp \cdot \nabla \bar{q}}{q_2 u_\perp^2} \mathbf{u}_\perp \right], \quad (\text{B.7})$$

where q is the potential vorticity and B is the Bernoulli function. The generalized entropies are $S[\bar{\omega}, h] = -\int C(\bar{\omega}) h d\mathbf{r}$. If we apply the anticipated vorticity method of Sec. 6, we get $\mathbf{J}_\omega = -(D/u^2)(\mathbf{u} \cdot \nabla \bar{q})\mathbf{u}$ which is equivalent to Eq. (B.5).

Appendix C. Local conservation of angular momentum

The equations derived in Sections 5 and 6 conserve the energy locally, but they do not conserve the angular momentum. For the sake of completeness, using similar methods, we derive here an equation that conserves the angular momentum locally. However, this equation is of little practical interest since it does not conserve the energy^{††}.

Appendix C.1. From the MEPP

The conservation of angular momentum $L = \int \bar{\omega} r^2 d\mathbf{r}$ leads to the global constraint $\dot{L} = 2 \int \mathbf{J}_\omega \cdot \mathbf{r} d\mathbf{r} = 0$. Applying the MEPP with the local constraint $\mathbf{J}_\omega \cdot \mathbf{r} = 0$, and writing the variational problem in the form

$$\begin{aligned} \delta \dot{S} - \int \frac{\lambda(\mathbf{r}, t)}{2} \delta(\mathbf{J}_\omega \cdot \mathbf{r}) d\mathbf{r} - \int \zeta(\mathbf{r}, t) \cdot \delta \left(\int \mathbf{J} d\sigma \right) d\mathbf{r} \\ - \int \frac{1}{D(\mathbf{r}, t)} \delta \left(\int \frac{\mathbf{J}^2}{2\rho} d\sigma \right) d\mathbf{r} = 0, \end{aligned} \quad (\text{C.1})$$

we obtain the optimal current

$$\mathbf{J} = -D(\mathbf{r}, t) [\nabla \rho + \lambda(\mathbf{r}, t) \rho(\sigma - \bar{\omega}) \mathbf{r}]. \quad (\text{C.2})$$

The Lagrange multiplier ζ has been eliminated, using the condition $\int \mathbf{J} d\sigma = \mathbf{0}$ of local normalization conservation. The vorticity current is

$$\mathbf{J}_\omega = -D(\mathbf{r}, t) [\nabla \bar{\omega} + \lambda(\mathbf{r}, t) \omega_2 \mathbf{r}]. \quad (\text{C.3})$$

^{††}It is not possible to conserve locally the angular momentum and the energy since the vorticity current \mathbf{J}_ω cannot be perpendicular to both $\nabla \psi$ and \mathbf{r} when these two vectors are not colinear.

The evolution of the Lagrange multiplier $\lambda(\mathbf{r}, t)$ is determined by introducing Eq. (C.3) in the local constraint $\mathbf{J}_\omega \cdot \mathbf{r} = 0$. This yields $\mathbf{r} \cdot \nabla \bar{\omega} + \lambda \omega_2 r^2 = 0$ implying

$$\lambda(\mathbf{r}, t) = -\frac{\mathbf{r} \cdot \nabla \bar{\omega}}{\omega_2 r^2}. \quad (\text{C.4})$$

Finally, we obtain the equations

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = \nabla \cdot \left\{ D(\mathbf{r}, t) \left[\nabla \rho - \rho(\sigma - \bar{\omega}) \frac{\mathbf{r} \cdot \nabla \bar{\omega}}{\omega_2 r^2} \mathbf{r} \right] \right\} \quad (\text{C.5})$$

and

$$\frac{\partial \bar{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \bar{\omega} = \nabla \cdot \left\{ D(\mathbf{r}, t) \left[\nabla \bar{\omega} - \frac{\mathbf{r} \cdot \nabla \bar{\omega}}{r^2} \mathbf{r} \right] \right\} = \frac{\partial}{\partial \theta} \left(\frac{D}{r^2} \frac{\partial \bar{\omega}}{\partial \theta} \right). \quad (\text{C.6})$$

Since the diffusion coefficient is unspecified, we can take $D \propto r^2$ to avoid dividing by zero when $r = 0$. Proceeding as in Section 5, we can show that Eq. (C.5) conserves the normalization, the Casimirs, the angular momentum, and increases monotonically the mixing entropy. Furthermore, Eq. (C.6) dissipates all the generalized enstrophies monotonically. The vorticity current vanishes for any axisymmetric flow $\bar{\omega} = f(r)$. Starting from a non-axisymmetric initial condition, this equation is expected to reach a steady state of the Euler equation that is axisymmetric but its precise form cannot be determined *a priori*. It depends on the dynamics and we have to solve Eq. (C.6).

Remark: if we take $D = Kr^2$ and ignore the advection term, the solution of Eq. (C.6) is $\bar{\omega}(r, \theta, t) = \sum_n \phi_n(r) e^{in\theta} e^{-Kn^2 t}$ where the $\phi_n(r)$ are determined by the initial condition $\bar{\omega}(r, \theta, 0)$. We find that $\bar{\omega}(r, \theta, t) \rightarrow \phi_0(r)$ for $t \rightarrow +\infty$ where $\phi_0(r) = \frac{1}{2\pi} \int_0^{2\pi} \bar{\omega}(r, \theta, 0) d\theta$. This is a trivial example which shows how the asymptotic state of Eq. (C.6) is selected.

Appendix C.2. From the anticipated vorticity method

We can proceed as in Section 6 to obtain an equation conserving locally the angular momentum while dissipating monotonically all the generalized enstrophies. We assume that the angular momentum is conserved locally so that $\mathbf{J}_\omega \cdot \mathbf{r} = 0$. This implies that $\mathbf{J}_\omega = -\lambda(\mathbf{r}, t) \mathbf{r}_\perp$ where $\lambda(\mathbf{r}, t)$ is an arbitrary function. Substituting this relation in Eq. (33) we get $\dot{S} = \int C''(\bar{\omega}) \lambda(\mathbf{r}, t) \mathbf{r}_\perp \cdot \nabla \bar{\omega} d\mathbf{r}$. If we take $\lambda(\mathbf{r}, t) = K(\mathbf{r}, t) \mathbf{r}_\perp \cdot \nabla \bar{\omega}$ with $K \geq 0$, we obtain $\dot{S} \geq 0$. Finally, it is relevant to write $K = D/r^2$ where $D(\mathbf{r}, t) \geq 0$ has the dimension of a diffusion coefficient. Therefore $\mathbf{J}_\omega = -D(\mathbf{r}_\perp \cdot \nabla \bar{\omega}) \mathbf{r}_\perp / r^2$. Substituting this expression in Eq. (13), we obtain

$$\frac{\partial \bar{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \bar{\omega} = \nabla \cdot \left[D \frac{(\mathbf{r}_\perp \cdot \nabla \bar{\omega}) \mathbf{r}_\perp}{r^2} \right] = \frac{\partial}{\partial \theta} \left(\frac{D}{r^2} \frac{\partial \bar{\omega}}{\partial \theta} \right). \quad (\text{C.7})$$

This equation can also be written as

$$\frac{\partial \bar{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \bar{\omega} = \nabla \cdot \left(D \frac{\mathbf{r}_\perp \otimes \mathbf{r}_\perp}{r^2} \nabla \bar{\omega} \right). \quad (\text{C.8})$$

Since D is unspecified, we can take $D = Kr^2$ in order to avoid dividing by zero when $r = 0$. This equation conserves locally the angular momentum and decreases monotonically all the generalized enstrophies. Indeed

$$\dot{S} = \int C''(\bar{\omega}) \frac{D}{r^2} (\mathbf{r}_\perp \cdot \nabla \bar{\omega})^2 d\mathbf{r} \geq 0. \quad (\text{C.9})$$

The diffusion current vanishes for any axisymmetric flow $\bar{\omega} = f(r^2)$. Actually, Eq. (C.7) is equivalent to Eq. (C.6) derived from the MEPP. In the present case, this is obvious in view of the last equality in Eqs. (C.6) and (C.7) but we can also show it by making the parallel with the calculations of Section 6. Combining the identity of vector analysis

$$\mathbf{r}_\perp \times (\nabla \bar{\omega} \times \mathbf{r}_\perp) = r^2 \nabla \bar{\omega} - (\mathbf{r}_\perp \cdot \nabla \bar{\omega}) \mathbf{r}_\perp \quad (\text{C.10})$$

with the relation

$$\nabla \bar{\omega} \times \mathbf{r}_\perp = \nabla \bar{\omega} \times (\mathbf{z} \times \mathbf{r}) = (\nabla \bar{\omega} \cdot \mathbf{r}) \mathbf{z} \quad (\text{C.11})$$

leading to

$$\mathbf{r}_\perp \times (\nabla \bar{\omega} \times \mathbf{r}_\perp) = (\nabla \bar{\omega} \cdot \mathbf{r}) \mathbf{r}, \quad (\text{C.12})$$

we find that

$$\frac{(\mathbf{r}_\perp \cdot \nabla \bar{\omega}) \mathbf{r}_\perp}{r^2} = \nabla \bar{\omega} - \frac{(\nabla \bar{\omega} \cdot \mathbf{r}) \mathbf{r}}{r^2}. \quad (\text{C.13})$$

Therefore, Eq. (C.7) is equivalent to Eq. (C.6).

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